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AN ANALYSIS OF LARGE-STRAIN DAMAGE ELASTOPLASTICITY

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Abstract—The elastoplastic constitutive analysis which utilizes the model of multiplicative decomposition of the deformation gradient into its elastic and plastic parts has been mainly developed and applied to elastically isotropic materials, which remain isotropic during the process of plastic deformation. This paper extends the application of the model to materials that change their elastic properties during the deformation process as a result of the material degradation and the corresponding damage. The exact kinematic and kinetic analysis of the finite deformations leads to an additive decomposition of the total strain rate into its elastic, damage and plastic constituents. The general structure of the expression for the damage strain rate is derived, valid for utilized damage tensors of any order. The analysis of elastoplastic deformation of elastically anisotropic materials without damage is also presented, with the application to transversely isotropic materials. The relationships between the elastic and plastic strain rates and the components of the multiplicative decomposition and their rates are also given.

1. INTRODUCTION

Let $\mathscr{B}_0$ be the initial, undeformed configuration of a considered material sample, and $\mathscr{B}_t$ its deformed configuration obtained by a specified loading program from the initial to current time t. Assume that a loading is beyond the elastic limit, so that inelastic deformation processes take place, pertinent to internal structure and composition of the considered material. For example, if the material is a ductile metal, inelasticity is caused by the dislocation motion and related micromechanisms occurring within a metal polycrystalline structure. For a brittle material, such as rock or concrete, inelastic deformation is a consequence of the evolution of internal crack structure, i.e. the initiation and propagation of microfractures within the material sample. Whatever the cause of inelasticity is, let $\mathbf{F}$ be the deformation gradient that maps the infinitesimal material element $d\mathbf{X}$ from its initial configuration to its current configuration $d\mathbf{x}$, i.e. $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Both the initial $\mathbf{X}$ and the current $\mathbf{x}$ locations of the material particle are referred to the same, fixed set of the rectangular coordinate axes. Introduce next the intermediate reference configuration $\mathscr{P}_t$ by elastic distressing the current configuration $\mathscr{B}_t$ to zero stress. Therefore, defined configuration differs from the initial configuration by residual (plastic) deformation, and from the current configuration by reversible (elastic) deformation. If $d\mathbf{p}$ is the material element in $\mathscr{P}_t$, corresponding to its configuration $d\mathbf{x}$ in $\mathscr{B}_t$, then $d\mathbf{x} = \mathbf{F}_e d\mathbf{p}$, where $\mathbf{F}_e$ denotes the deformation gradient associated with elastic loading from $\mathscr{P}_t$ to $\mathscr{B}_t$. Introducing also the deformation gradient of the transformation $\mathscr{B}_0 \rightarrow \mathscr{P}_t$, by $d\mathbf{p} = \mathbf{F}_p d\mathbf{X}$, the multiplicative decomposition of deformation gradient follows (Lee, 1969):

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p. \tag{1}$$

$\mathbf{F}_e$ is customarily called elastic, and $\mathbf{F}_p$ plastic part of the total deformation gradient $\mathbf{F}$. For inhomogeneous deformations, only $\mathbf{F}$ is the true deformation gradient, whose components are the partial derivatives $\partial\mathbf{x}/\partial\mathbf{X}$. In contrast, the mappings $\mathscr{P}_t \rightarrow \mathscr{B}_t$ and $\mathscr{B}_0 \rightarrow \mathscr{P}_t$ are not, in general, continuous one-to-one mappings, so that $\mathbf{F}_e$ and $\mathbf{F}_p$ are not defined as the gradients of the respective mappings (which may not exist), but as the point functions (local deformation gradients). In the case when elastic distressing to zero stress ($\mathscr{B}_t \rightarrow \mathscr{P}_t$) is not physically achievable due to the onset of reverse inelastic deformation before the zero stress is reached (which often occurs at advanced stages $\mathscr{B}_t$ of deformation due to anisotropic hardening and strong Bauschinger effects in ductile metals, or due to the incomplete frictional back-sliding of the crack faces in brittle rocks), the intermediate configuration can be conceptually introduced by virtual distressing to zero stress, locking all inelastic structural changes that would occur during the actual distressing.

Deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_p$ are not uniquely defined, because arbitrary local material element rotations superposed to unstressed state give alternate intermediate configurations. However, if the material is elastically isotropic and remains such during the inelastic deformation, preserving its elastic properties, the elastic strain energy ψ_e per unit unstressed volume is an isotropic function of the right Cauchy–Green elastic deformation tensor $\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e$, i.e. $\psi_e(\mathbf{Q}\mathbf{C}_e\mathbf{Q}^T) = \psi_e(\mathbf{C}_e)$. Here, $\mathbf{Q}$ is an orthogonal tensor corresponding to arbitrary rigid-body rotation superposed to the unstressed state (superscript T denotes the transpose). The elastic stress response from $\mathscr{P}_t \rightarrow \mathscr{B}_t$ is, therefore, not influenced by the nonuniqueness of intermediate configuration and is given by the well-known isotropic finite elasticity law (Truesdell and Noll, 1965)

$$\boldsymbol{\sigma} = \frac{2}{|\mathbf{F}_e|}\mathbf{B}_e\frac{\partial\psi_e(\mathbf{B}_e)}{\partial\mathbf{B}_e}. \tag{2}$$

In eqn (2), the strain energy (per unit unstressed volume) ψ_e is an isotropic function of the left Cauchy–Green elastic deformation tensor $\mathbf{B}_e = \mathbf{F}_e\mathbf{F}_e^T$, $|\ |$ denotes the determinant and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. This structure of elasticity law was used in a series of papers on the elastoplastic constitutive equations by Lee and his coworkers (Lee, 1969; Lubarda and Lee, 1981; Agah-Tehrani *et al.*, 1987), by Lubarda (1991a, 1994) and Lubarda and Shih (1994).

Few attempts were made to extend the analysis based on the multiplicative decomposition to materials that are elastically anisotropic in its initial (underformed) configuration, or to materials that develop elastic anisotropy during a course of inelastic deformation (Dafalias, 1985; Lubarda, 1991b). In fact, since in most elaborations it was assumed that elastic properties are not influenced by the previous inelastic processes, which is an unacceptable assumption in many cases of engineering importance, the usefulness of the decomposition was seriously questioned (Nemat-Nasser, 1982). The difficulty was partly related to the nonuniqueness of the unstressed configuration, its consequences on the anisotropic elastic response, and anticipated mathematical difficulties that may arise in proper handling of the analysis. This paper is, consequently, devoted to the generalization of the existing constitutive analysis, based on the multiplicative decomposition, to materials that change their elastic properties during the inelastic deformation process, and exhibit the damage–elastoplastic response. The general formulation is presented, restricted to isothermal and time-independent material behavior.

2. DESCRIPTION OF ANISOTROPIC ELASTIC RESPONSE

Consider an intermediate configuration $\mathscr{P}_t$ obtained by distressing the current configuration $\mathscr{B}_t$ to zero stress. Assume that the material in configuration $\mathscr{P}_t$ is elastically anisotropic, either because it was initially anisotropic, or because it has developed elastic anisotropy during the previous inelastic deformation (for example, due to the grain rotations in a polycrystalline metal sample and the consequent crystallographic texture, or due to anisotropic crack progression in the brittle rock samples). Therefore, let $\mathscr{D}$ denote a set of the symmetric tensor variables of various orders (scalars, second-order, fourth-order tensors, etc.), attached to the current configuration $\mathscr{B}_t$, which appropriately account for the degradation of elastic material properties and their directional changes, accumulated during the previous inelastic deformation. The variables $\mathscr{D}$ will be referred to as the damage variables. For example, in modeling inelastic behavior with infinitesimal elastic component of strain, the current (degraded) fourth-order elastic stiffness tensor can be selected as an appropriate damage tensor (Dougill, 1983). Ortiz (1985) has used the current elastic compliance tensor as the damage tensor in his study of inelastic behavior of concrete [see also Lubarda and Krajcinovic (1993, 1994a)].

Even if there is no degradation of elastic material properties, the tensor variables $\mathscr{D}$ can be introduced to properly and conveniently describe the state of initial elastic anisotropy of the material [structural tensors (Boehler, 1987)]. For example, in the case of transverse isotropy with the axis of isotropy in the current configuration $\mathscr{B}_t$ coincident with the direction $\mathbf{n}$, the structural tensor is the second-order tensor $\mathscr{D} = \mathbf{n} \otimes \mathbf{n}$. For orthotropic material with the principal directions of orthotropy coincident with the directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, the structural tensors are $\mathbf{n}_1 \otimes \mathbf{n}_1$, $\mathbf{n}_2 \otimes \mathbf{n}_2$ and $\mathbf{n}_3 \otimes \mathbf{n}_3 = \mathbf{I} - \mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2$ ($\mathbf{I}$ denotes the second-order unit tensor and $\otimes$ the outside tensor product). The structural tensors corresponding to a general elastic anisotropy can be similarly formed and are given in Boehler (1987).

The introduced damage variables can only change during continuing inelastic deformation but remain unaltered during elastic unloading or reverse elastic loading, except for the elastic embedding which convects them together with the material. Therefore, the damage variables $\mathscr{D}$ in the current configuration $\mathscr{B}_t$ become the variables $\hat{\mathscr{D}}$ in the intermediate configuration, induced from $\mathscr{D}$ by elastic deformation $\mathbf{F}_e$. For example, for the second-order damage tensor, the induced tensor can be defined by a transformation of the weighted contravariant or covariant type, i.e.

$$\hat{\mathscr{D}} = |\mathbf{F}_e|^m \mathbf{F}_e^{-1} \mathscr{D} \mathbf{F}_e^{-T} \quad \text{or} \quad \hat{\mathscr{D}} = |\mathbf{F}_e|^{-m} \mathbf{F}_e^{T} \mathscr{D} \mathbf{F}_e, \tag{3}$$

where m is the weight and (-1) denotes the inverse. For the fourth-order damage tensor, the corresponding induced tensor is

$$\hat{\mathscr{D}} = |\mathbf{F}_e|^m \mathbf{F}_e^{-1} \otimes \mathbf{F}_e^{-1} \mathscr{D} \mathbf{F}_e^{-T} \otimes \mathbf{F}_e^{-T} \quad \text{or} \quad \hat{\mathscr{D}} = |\mathbf{F}_e|^{-m} \mathbf{F}_e^{T} \otimes \mathbf{F}_e^{T} \mathscr{D} \mathbf{F}_e \otimes \mathbf{F}_e. \tag{4}$$

For example, the second tensor in eqn (4) has the components

$$\hat{\mathscr{D}}_{ijkl} = |\mathbf{F}_e|^{-m} (F^e_{i\alpha})^T (F^e_{j\beta})^T \mathscr{D}_{\alpha\beta\gamma\delta} F^e_{\gamma k} F^e_{\delta l}. \tag{5}$$

To describe the elastic response of anisotropic material at the current state of deformation and material damage, the strain energy ψ per unit initial volume is assumed to be given by

$$\psi = \psi(\mathbf{C}_e, \hat{\mathscr{D}}). \tag{6}$$

Note that $\psi = |\mathbf{F}_p| \psi_e$, where ψ_e is the elastic strain energy per unit unstressed volume in the intermediate configuration. Since the unit of unstressed volume contains a varying amount of mass during the deformation process whenever plastic deformation is compressible, so that $|\mathbf{F}_p| \neq 1$, the strain energy ψ per unit initial volume is introduced in eqn (6), as it always refers to a fixed amount of mass.

Since the material response is independent of the superposed rotation to intermediate-unstressed configuration, eqn (6) has to be an isotropic scalar function of the set of all its arguments, i.e. $\mathbf{C}_e$ and $\hat{\mathscr{D}}$. For example, if the set $\mathscr{D}$ consists of the second-order tensors $\mathscr{D}_2$ and the fourth-order tensors $\mathscr{D}_4$, the isotropy of ψ requires that for every orthogonal transformation $\mathbf{Q}$,

$$\psi(\mathbf{Q}\mathbf{C}_e\mathbf{Q}^T, \mathbf{Q}\hat{\mathscr{D}}_2\mathbf{Q}^T, \mathbf{Q} \otimes \mathbf{Q} \hat{\mathscr{D}}_4 \mathbf{Q}^T \otimes \mathbf{Q}^T) = \psi(\mathbf{C}_e, \hat{\mathscr{D}}_2, \hat{\mathscr{D}}_4). \tag{7}$$

Note that under the superposed rotation $\mathbf{Q}$ of the intermediate configuration, $\mathscr{D}$ does not change, as it is defined with respect to the current configuration. Since $\mathbf{F}_e$ changes to $\mathbf{F}_e\mathbf{Q}^T$, from eqn (3) it follows that $\hat{\mathscr{D}}_2$ changes to $\mathbf{Q}\hat{\mathscr{D}}_2\mathbf{Q}^T$. An analogous change rule applies to the fourth-order damage tensor $\hat{\mathscr{D}}_4$, as utilized in eqn (7).

The theory of isotropic scalar and tensor functions of several tensor arguments has been extensively studied in the literature. A comprehensive treatment of various important issues has been presented by Spencer (1971, 1987) and Boehler (1977, 1987). There the integrity basis for the considered functions are derived mainly for the vector and second-order tensor arguments. Betten (1982, 1987, 1992) has also considered the functions that depend on the second- and fourth-order tensors and construction of their individual and joint invariants. For example, if $\hat{\mathscr{D}}$ in eqn (6) is a single second-order symmetric tensor, ψ can be represented as a polynomial of its irreducible integrity basis consisting of the following invariants:

$$(\mathbf{C}_e : \mathbf{I}), (\mathbf{C}_e : \mathbf{C}_e), (\mathbf{C}_e^2 : \mathbf{C}_e), (\hat{\mathscr{D}} : \mathbf{I}), (\hat{\mathscr{D}} : \hat{\mathscr{D}})$$

$$(\hat{\mathscr{D}}^2 : \hat{\mathscr{D}}), (\mathbf{C}_e : \hat{\mathscr{D}}), (\mathbf{C}_e : \hat{\mathscr{D}}^2), (\mathbf{C}_e^2 : \hat{\mathscr{D}}), (\mathbf{C}_e^2 : \hat{\mathscr{D}}^2). \tag{8}$$

In eqn (8), (:) stands for the inner (trace) product of the second-order tensors. The integrity basis can be written for any finite set of second-order tensors. Spencer (1971) provides a list of invariants and the integrity bases for the polynomial scalar functions dependent on one to six second-order tensor arguments. For general (not necessarily polynomial) functions, the integrity bases are replaced by the function bases, which, in general, contain fewer terms than the corresponding integrity bases. For example, the function bases of the general scalar function dependent on an arbitrary number of second-order tensors are composed of the traces of the products of all unordered combinations of only one, two and three tensorial arguments (Boehler, 1977).

The construction of the integrity bases for the second- and fourth-order symmetric tensors is a more difficult task. Some of the individual and joint invariants are listed below [for the more complete list, refer to Betten (1987, 1992)]:

$$\hat{\mathscr{D}} :: (\mathbf{I} \otimes \mathbf{I}), (\hat{\mathscr{D}} :: \mathbf{II}), (\hat{\mathscr{D}} :: \hat{\mathscr{D}})$$

$$\hat{\mathscr{D}} :: (\mathbf{C}_e \otimes \mathbf{C}_e), (\mathbf{I} : \hat{\mathscr{D}} : \mathbf{C}_e^2), (\mathbf{C}_e : \hat{\mathscr{D}}) : (\hat{\mathscr{D}} : \mathbf{C}_e). \tag{9}$$

In eqn (9), **II** is the fourth-order unit tensor, while :: designates the trace, so that for the two fourth-order tensors **A** and **B**, $\mathbf{A} :: \mathbf{B} = A_{ijkl}B_{ijkl}$. The trace of the fourth-order tensor **A** and the second-order tensor **C** is the second-order tensor $\mathbf{A} : \mathbf{C}$, with the components $A_{ijkl}C_{kl}$.

In general, the stress response from the intermediate to current configuration is given by

$$\boldsymbol{\sigma} = \frac{2}{|\mathbf{F}_e|}\mathbf{F}_e \frac{\partial \psi_e}{\partial \mathbf{C}_e} \mathbf{F}_e^T, \tag{10}$$

which is independent of the rigid-body rotation superposed to the intermediate configuration. This clearly follows since $\boldsymbol{\sigma}$ does not change under the superposed rotation **Q** of the intermediate configuration, while $\mathbf{F}_e$ changes to $\mathbf{F}_e\mathbf{Q}^T$, and $\mathbf{C}_e$ to $\mathbf{Q}\mathbf{C}_e\mathbf{Q}^T$. More detailed discussion of the objectivity issues in the formulation of elastoplasticity theory by using the multiplicative decomposition of the deformation gradient is presented by Lubarda (1991a).

Expressing the strain energy ψ_e per unit unstressed volume in terms of the strain energy ψ per unit initial volume, eqn (10) can be rewritten as

$$\boldsymbol{\tau} = 2\mathbf{F}_e \frac{\partial \psi(\mathbf{C}_e, \hat{\mathscr{D}})}{\partial \mathbf{C}_e} \mathbf{F}_e^T, \tag{11}$$

where $\boldsymbol{\tau} = |\mathbf{F}|\boldsymbol{\sigma}$ is the Kirchhoff stress and $\psi = |\mathbf{F}_p|\psi_e$ is the strain energy per unit initial volume. In the rate-type constitutive analysis considered in this paper, it will be useful to start the analysis with the finite elasticity law [eqn (11)], even when intended application is to material behavior with infinitesimal elastic components of strain.

3. RATE-TYPE ANALYSIS

To derive the rate-type constitutive equations of the damage–elastoplastic material behavior, apply first the material time derivative (designated by the superimposed dot) to both sides of eqn (11). By an appropriate and straightforward rearrangement of the terms, it follows that

$$\dot{\boldsymbol{\tau}} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})\boldsymbol{\tau}+\boldsymbol{\tau}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})^{\mathrm{T}}+2\mathbf{F}_e\left(\frac{\partial^2\psi}{\partial\mathbf{C}_e\otimes\partial\mathbf{C}_e}:\dot{\mathbf{C}}_e\right)\mathbf{F}_e^{\mathrm{T}}+2\mathbf{F}_e\left(\frac{\partial^2\psi}{\partial\mathbf{C}_e\otimes\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}}\right)\mathbf{F}_e^{\mathrm{T}}. \tag{12}$$

In view of eqn (11), which gives the stress $\boldsymbol{\tau}$ as a function of $\mathbf{F}_e$ and $\hat{\mathscr{D}}$, the last term on the right-hand side of eqn (12) can be written in a compact form as

$$2\mathbf{F}_e\left(\frac{\partial^2\psi}{\partial\mathbf{C}_e\otimes\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}}\right)\mathbf{F}_e^{\mathrm{T}} = \frac{\partial\boldsymbol{\tau}}{\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}}. \tag{13}$$

In eqn (13), $\partial\boldsymbol{\tau}/\partial\hat{\mathscr{D}}$ designates the partial derivative of the stress expression (11) with respect to $\hat{\mathscr{D}}$, at constant $\mathbf{F}_e$. Further, since

$$\dot{\mathbf{C}}_e = 2\mathbf{F}_e^{\mathrm{T}}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s\mathbf{F}_e, \tag{14}$$

where the subscript s designates the symmetric part, the third term on the right-hand side of eqn (12) can be written as

$$2\mathbf{F}_e\left(\frac{\partial^2\psi}{\partial\mathbf{C}_e\otimes\partial\mathbf{C}_e}:\dot{\mathbf{C}}_e\right)\mathbf{F}_e^{\mathrm{T}} = \boldsymbol{\Lambda}_e:(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s. \tag{15}$$

Here, $\boldsymbol{\Lambda}_e$ is the fourth-order tensor with the rectangular components

$$\Lambda^e_{ijkl} = 4\,F^e_{im}F^e_{jn}\frac{\partial^2\psi}{\partial C^e_{mn}\partial C^e_{pq}}F^e_{kp}F^e_{lq}. \tag{16}$$

Substitution of eqns (13) and (15) into eqn (12) therefore gives

$$\dot{\boldsymbol{\tau}} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})\boldsymbol{\tau}+\boldsymbol{\tau}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})^{\mathrm{T}}+\boldsymbol{\Lambda}_e:(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s+\frac{\partial\boldsymbol{\tau}}{\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}}. \tag{17}$$

To proceed further with the rate-type constitutive analysis, consider the velocity gradient in the current configuration at time t, defined by $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Introducing the multiplicative decomposition of the deformation gradient [eqn (1)], the velocity gradient can be expressed as

$$\mathbf{L} = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1}+\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}. \tag{18}$$

The strain rate $\mathbf{D}$ and the spin $\mathbf{W}$ are given by the symmetric and antisymmetric parts of $\mathbf{L}$:

$$\mathbf{D} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s+[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s \tag{19}$$

$$\mathbf{W} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_a+[\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a. \tag{20}$$

Writing $\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}$ as the sum of its symmetric and antisymmetric parts, and using eqn (20) to express the antisymmetric part, one has

$$\dot{\mathbf{F}}_e\mathbf{F}_e^{-1} = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s + \mathbf{W} - \boldsymbol{\omega}. \tag{21}$$

For convenience, the tensor $\boldsymbol{\omega}$ in eqn (21) denotes the spin

$$\boldsymbol{\omega} = [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_a. \tag{22}$$

Substitution of eqn (21) into eqn (17) consequently gives

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_e : (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s + \frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathscr{D}}} : \dot{\hat{\mathscr{D}}} + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}, \tag{23}$$

where

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W} \tag{24}$$

represents the Jaumann derivative of the Kirchhoff stress $\boldsymbol{\tau}$. The material derivative of the Kirchhoff stress appearing on the right-hand side of eqn (24) is $\dot{\boldsymbol{\tau}} = |\mathbf{F}|(\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\ \mathrm{tr}\mathbf{D})$, where tr denotes the trace.

The fourth-order tensor of the instantaneous elastic moduli $\mathscr{L}_e$, appearing in eqn (23), has the rectangular components given by

$$\mathscr{L}^e_{ijkl} = \tfrac{1}{2}(\delta_{ik}\tau_{jl} + \delta_{il}\tau_{jk} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}) + 4F^e_{im}F^e_{jn}\frac{\partial^2\psi}{\partial C^e_{mn}\partial C^e_{pq}}F^e_{kp}F^e_{lq}, \tag{25}$$

where δ_{ij} denotes the Kronecker delta. In view of the introduced isotropy of the strain energy function ψ, it is easily shown that the components [eqn (25)] are independent of the superposed rotation of the intermediate configuration. In other words, any one from infinitely many by rotation differing deformation gradients $\mathbf{F}_e$ when substituted into eqn (25), gives the same values of the instantaneous elastic moduli. In metals the elastic moduli are usually far greater than the applied stresses and the two fourth-order tensors, whose components are given by eqns (16) and (25), are approximately equal to each other.

4. THE RATES OF DAMAGE TENSORS

Consider first $\hat{\mathscr{D}}$ to be the second-order damage tensor. The material time derivative of the induced tensor of the contravariant type, $\hat{\mathscr{D}} = |\mathbf{F}_e|^m\mathbf{F}_e^{-1}\mathscr{D}\mathbf{F}_e^{-T}$, is

$$\dot{\hat{\mathscr{D}}} = |\mathbf{F}_e|^m\mathbf{F}_e^{-1}\overset{*e}{\mathscr{D}}\mathbf{F}_e^{-T}, \tag{26}$$

where

$$\overset{*e}{\mathscr{D}} = \dot{\mathscr{D}} - (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})\mathscr{D} - \mathscr{D}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})^T + m\mathscr{D}\,\mathrm{tr}\,(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}) \tag{27}$$

is the corresponding (Oldroyd/Truesdell type) convected derivative, relative to the velocity gradient $\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}$. If the induced tensor of the covariant type is used, i.e. $\hat{\mathscr{D}} = |\mathbf{F}_e|^{-m}\mathbf{F}_e^T\mathscr{D}\mathbf{F}_e$, one has

$$\dot{\hat{\mathscr{D}}} = |\mathbf{F}_e|^{-m}\mathbf{F}_e^T\overset{*e}{\mathscr{D}}\mathbf{F}_e^{-T}, \tag{28}$$

where

$$\overset{\bullet e}{\mathscr{D}} = \dot{\mathscr{D}} + \mathscr{D}(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}) + (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})^T\mathscr{D} - m\mathscr{D}\,\mathrm{tr}\,(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}) \tag{29}$$

is the corresponding (Cotter–Rivlin/Truesdell type) convected derivative, associated with the covariant transformation. Substitution of either eqn (26) or (28) into the second term on the right-hand side of expression (23) therefore gives

$$\frac{\partial\tau}{\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}} = \frac{\partial\tau}{\partial\mathscr{D}}:\overset{\bullet e}{\mathscr{D}}. \tag{30}$$

It should be observed that the introduced convected derivatives [eqns (27) and (29)] are not uniquely defined because the unstressed intermediate configuration is specified only to within an arbitrary rigid-body rotation, so that the velocity gradient $\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}$, used in eqns (27) and (29), is not uniquely defined either. However, in some applications it may be convenient to specify the intermediate configuration uniquely, on the basis of some additional physical structure, explicitly introduced in the considered material model and pertinent to its internal structure and the deformation modes. For example, in the crystal plasticity (Asaro, 1983), the rotation of the intermediate configuration is uniquely specified by requiring that the basic crystalline (lattice) structure always has the same orientation relative to the fixed reference frame [isoclinic intermediate configuration, in the terminology of Mandel (1971, 1973)]. In this case, the velocity gradient $\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}$ is uniquely defined and represents the sum of the lattice strain rate and the lattice spin. Physically, it is the discontinuous slip of the material over the active slip planes that causes the lattice orientation to be convected by the lattice and not by the material itself.

On the other hand, in some applications it may be more appropriate to introduce the convected derivative as the derivative observed in the reference frame that deforms with the material, i.e. relative to the material velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. For example, in brittle materials like brittle rocks, the change of elastic properties occurs due to propagation of the crack-like defects through the material, which convects them with itself during the deformation process. Therefore, by using eqn (18) to eliminate $(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})$, eqn (26) can be rewritten as

$$\dot{\hat{\mathscr{D}}} = |\mathbf{F}_e|^m\mathbf{F}_e^{-1}\overset{\circ}{\mathscr{D}}\mathbf{F}_e^{-T} + (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\hat{\mathscr{D}} + \hat{\mathscr{D}}(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})^T - m\hat{\mathscr{D}}\,\mathrm{tr}\,(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}), \tag{31}$$

where

$$\overset{\circ}{\mathscr{D}} = \dot{\mathscr{D}} - \mathbf{L}\mathscr{D} - \mathscr{D}\mathbf{L}^T + m\mathscr{D}\,\mathrm{tr}\mathbf{L}. \tag{32}$$

Similarly, eqn (28) can be rearranged as

$$\dot{\hat{\mathscr{D}}} = |\mathbf{F}_e|^{-m}\mathbf{F}_e^T\overset{*}{\mathscr{D}}\mathbf{F}_e - \hat{\mathscr{D}}(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}) - (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})^T\hat{\mathscr{D}} + m\hat{\mathscr{D}}\,\mathrm{tr}(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}), \tag{33}$$

where

$$\overset{*}{\mathscr{D}} = \dot{\mathscr{D}} + \mathscr{D}\mathbf{L} + \mathbf{L}^T\mathscr{D} - m\mathscr{D}\,\mathrm{tr}\mathbf{L}. \tag{34}$$

Substitution of either eqn (31) or (33) into the second term on the right-hand side of expression (23) now gives

$$\frac{\partial\tau}{\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}} = \frac{\partial\tau}{\partial\mathscr{D}}:\overset{*}{\mathscr{D}} + \mathscr{A}:(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}). \tag{35}$$

When eqn (31) is used, the fourth-order tensor $\mathscr{A}$ is defined by

$$\mathscr{A}:(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}) = \frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathscr{D}}}:[(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})\hat{\mathscr{D}} + \hat{\mathscr{D}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})^{\mathrm{T}} - m\hat{\mathscr{D}}\,\mathrm{tr}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})]. \tag{36a}$$

If eqn (33) is used, eqn (36a) is replaced by

$$\mathscr{A}:(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}) = -\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathscr{D}}}:[\hat{\mathscr{D}}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}) + (\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})^{\mathrm{T}}\hat{\mathscr{D}} - m\hat{\mathscr{D}}\,\mathrm{tr}(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1})]. \tag{36b}$$

It should be pointed out that, although in this section the damage tensors were assumed to be the second-order tensors, the structure of expressions (30)–(35) remain the same for any introduced higher-order symmetric damage tensors. This is illustrated for the fourth-order tensors in the Appendix of this paper.

5. IDENTIFICATION OF ELASTIC AND DAMAGE STRAIN RATES

Substituting expression (30) into eqn (23) it follows

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_{\mathrm{e}}:(\dot{\mathbf{F}}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} + \frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}}:\overset{*\mathrm{e}}{\mathscr{D}} + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}. \tag{37}$$

If eqn (35) rather than eqn (30) is used, one has

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_{\mathrm{e}}:(\dot{\mathbf{F}}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} + \frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}}:\overset{*}{\mathscr{D}} + \mathscr{A}:(\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}) + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}. \tag{38}$$

Concentrating attention first to the case when eqn (37) applies, it is next shown that the strain rate $(\dot{\mathbf{F}}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}}$ consists of three parts: elastic strain rate $\mathbf{D}_{\mathrm{e}}$, damage strain rate $\mathbf{D}_{\mathrm{d}}$ and an additional part denoted by $\boldsymbol{\Delta}$, such that

$$(\dot{\mathbf{F}}_{\mathrm{e}}\mathbf{F}_{\mathrm{e}}^{-1})_{\mathrm{s}} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{d}} + \boldsymbol{\Delta}. \tag{39}$$

Indeed, substitution of eqn (39) into eqn (37) gives

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}_{\mathrm{e}}:\mathbf{D}_{\mathrm{e}} + \mathscr{L}_{\mathrm{e}}:\mathbf{D}_{\mathrm{d}} + \frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}}:\overset{*\mathrm{e}}{\mathscr{D}} + \mathscr{L}_{\mathrm{e}}:\boldsymbol{\Delta} + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}. \tag{40}$$

Since the instantaneous elastic moduli tensor $\mathscr{L}_{\mathrm{e}}$ and its inverse, the instantaneous elastic compliance tensor $\mathscr{L}_{\mathrm{e}}^{-1}$, possess required symmetry and reciprocity properties

$$(\mathscr{L}_{\mathrm{e}}^{-1})_{ijkl} = (\mathscr{L}_{\mathrm{e}}^{-1})_{ijlk} = (\mathscr{L}_{\mathrm{e}}^{-1})_{jikl} = (\mathscr{L}_{\mathrm{e}}^{-1})_{klij}, \tag{41}$$

it follows that $\mathscr{L}_{\mathrm{e}}^{-1}:\overset{\circ}{\boldsymbol{\tau}}$ is derivable from the elastic rate potential $\phi_{\mathrm{e}} = \frac{1}{2}\mathscr{L}_{\mathrm{e}}^{-1}:(\overset{\circ}{\boldsymbol{\tau}} \otimes \overset{\circ}{\boldsymbol{\tau}})$ as its gradient $\partial\phi_{\mathrm{e}}/\partial\overset{\circ}{\boldsymbol{\tau}}$. Therefore,

$$\mathbf{D}_{\mathrm{e}} = \mathscr{L}_{\mathrm{e}}^{-1}:\overset{\circ}{\boldsymbol{\tau}} \tag{42}$$

gives the reversible strain increment, that is recovered in a hardening material upon unloading of the Jaumann stress increment associated with $\overset{\circ}{\boldsymbol{\tau}}$. Further, it is natural to define the damage strain rate $\mathbf{D}_{\mathrm{d}}$ as the strain rate associated with progressive degradation of the material elastic properties, as represented by the change of damage variables $\mathscr{D}$. Hence, the damage strain rate $\mathbf{D}_{\mathrm{d}}$ has to be directly related to the rate of change tensor $\overset{*\mathrm{e}}{\mathscr{D}}$. Consequently, in view of the previously established relationship [eqn (42)] between $\mathbf{D}_{\mathrm{e}}$ and $\overset{\circ}{\boldsymbol{\tau}}$, it necessarily follows that eqn (40) splits into three equations:

$$\mathscr{L}_e : \mathbf{D}_e = \overset{\circ}{\boldsymbol{\tau}} \tag{43}$$

$$\mathscr{L}_e : \mathbf{D}_d + \frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}} : \overset{*e}{\mathscr{D}} = \mathbf{0} \tag{44}$$

$$\mathscr{L}_e : \boldsymbol{\Delta} + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau} = \mathbf{0}. \tag{45}$$

Inversion of eqn (43) gives the elastic strain rate expression (42). Solving eqn (44) for $\mathbf{D}_d$ provides the expression for the damage strain rate

$$\mathbf{D}_d = -\mathscr{L}_e^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}} : \overset{*e}{\mathscr{D}} \right) \tag{46}$$

With the evolution equation for the damage rate $\overset{*e}{\mathscr{D}}$ additionally constructed, eqn (46) explicitly gives the damage strain rate $\mathbf{D}_d$. Finally, condition (45) defines the remaining part of the strain rate $(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s$, appearing in eqn (39), i.e.

$$\boldsymbol{\Delta} = -\mathscr{L}_e^{-1} : (\boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}). \tag{47}$$

If eqn (38) is used in place of eqn (37), it is easily shown that the corresponding damage strain rate is defined by

$$\mathbf{D}_d = -\mathscr{L}_e^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}} : \overset{*}{\mathscr{D}} \right), \tag{48}$$

while expression (47) is replaced by

$$\boldsymbol{\Delta} = -\mathscr{L}_e^{-1} : [\mathscr{A} : (\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}) + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau}]. \tag{49}$$

It is clear that the right-hand side of eqn (48) is independent of the superposed rotation to the intermediate configuration, hence for the prescribed $\overset{*}{\mathscr{D}}$ and the known current state of the material, eqn (48) uniquely specifies the damage strain rate $\mathbf{D}_d$. Since $\mathbf{D}_e$ is also uniquely specified, the nonuniqueness of the strain rate $(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s$, associated with a possible superposed rigid-body rotation of the intermediate unstressed configuration, is all contained in the nonuniqueness of the $\boldsymbol{\Delta}$ part of this strain rate, given by eqn (49). However, in the subsequent analysis, the $\boldsymbol{\Delta}$ part of the strain rate is of no direct interest and its nonuniqueness does not present any problem.

6. PARTITION OF THE STRAIN RATE INTO ITS ELASTIC, DAMAGE AND PLASTIC PARTS

Substituting expression (39) for the strain rate $(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s$ into expression (18) for the total strain rate, it now follows that

$$\mathbf{D} = \mathbf{D}_e + \mathbf{D}_d + \boldsymbol{\Delta} + [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s. \tag{50}$$

Consequently, by defining the plastic strain rate $\mathbf{D}_p$ as

$$\mathbf{D}_p = [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s + \boldsymbol{\Delta}, \tag{51}$$

expression (50) gives the additive decomposition of the total strain rate into its elastic, damage and plastic constituents, i.e.

$$\mathbf{D} = \mathbf{D}_e + \mathbf{D}_d + \mathbf{D}_p. \tag{52}$$

For example, if the material behavior is such that the Ilyushin postulate can be adopted (Ilyushin, 1961), i.e. the nett work done during arbitrary closed strain cycle is positive, provided that inelastic deformation occurred during the cycle, it follows that $\mathbf{D}_p + \mathbf{D}_d = \mathbf{D} - \mathbf{D}_e$ part of the total strain rate is normal to the yield surface in stress space [see eqn (11) of Ilyushin (1961) and the discussion following eqn (16) of Hill (1968)]. If elastic properties are assumed to be uninfluenced by inelastic deformation, the damage strain rate is zero and $\mathbf{D}_p = \mathbf{D} - \mathbf{D}_e$ part of the strain rate is normal to the yield surface.

7. ELASTOPLASTICITY WITHOUT DAMAGE

As a special case of the general formulation presented in the previous sections, consider the elastoplastic deformation of an elastically anisotropic material without a damage. For the sake of simplicity, restrict attention to transversely isotropic material. A similar analysis can be performed in the case of elastic orthotropy or more general anisotropy. Let $\mathbf{n}_0$ be the unit vector parallel to the axis of isotropy in the initial undeformed configuration $\mathscr{B}_0$. In the current elastoplastically deformed configuration $\mathscr{B}_t$ the material is assumed to remain transversely isotropic, with the axis of isotropy parallel to the unit direction $\mathbf{n}$. The corresponding structural tensor is (Boehler, 1987)

$$\mathscr{D} = \mathbf{n} \otimes \mathbf{n}. \tag{53}$$

Due to discontinuous slip and other micromechanisms of plastic deformation, the direction of the isotropy axis is not, in general, convected by the total deformation gradient, i.e. $\mathbf{n} \neq |\mathbf{F}|^{-1}\mathbf{F}\mathbf{n}_0$. It will be convenient in this section to specify the intermediate unstressed configuration $\mathscr{P}_t$ to be isoclinic, so that the direction of the isotropy axis in $\mathscr{P}_t$ is parallel to its direction $\mathbf{n}_0$ in the initial undeformed configuration $\mathscr{B}_0$. Note that the so defined isoclinic configuration is unique to within an arbitrary rigid-body rotation about the axis of isotropy $\mathbf{n}_0$. Since the elastic material response from $\mathscr{P}_t$ to $\mathscr{B}_t$ is not influenced by the rotation about the isotropy axis, this rotation is of no further importance. If $\mathbf{F}_e$ is the deformation gradient from any of the introduced isoclinic configurations to the current configuration, it follows that

$$\mathbf{n} = |\mathbf{F}_e|^{-1}\,\mathbf{F}_e\mathbf{n}_0. \tag{54}$$

Equation (54) holds because the axis of isotropy can be considered to be embedded in the material during the elastic deformation $\mathbf{F}_e$. The induced structural tensor in the intermediate configuration is, therefore, obtained by the contravariant-type transformation

$$\hat{\mathscr{D}} = |\mathbf{F}_e|^2\,\mathbf{F}_e^{-1}\,\mathscr{D}\,\mathbf{F}_e^{-\mathrm{T}} = \mathbf{n}_0 \otimes \mathbf{n}_0. \tag{55}$$

Since $\hat{\mathscr{D}}$ is a constant tensor, from eqn (26) it follows that $\overset{*e}{\mathscr{D}} = \mathbf{0}$, and eqn (46) gives that the damage strain rate is also equal to zero, $\mathbf{D}_d = \mathbf{0}$. This was naturally expected to be the case, because it is assumed that the material remains transversely isotropic, with the unchanged elastic properties.

Consequently, expression (39) reduces to

$$(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s = \mathbf{D}_e + \mathbf{\Delta}. \tag{56}$$

Equation (56), together with expression (47) for the $\mathbf{\Delta}$ part of the strain rate, provides the explicit relationship between the elastic strain rate $\mathbf{D}_e$ and the constituents $\mathbf{F}_e$ and $\mathbf{F}_p$, and their rates, of the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$,

$$\mathbf{D}_e = (\dot{\mathbf{F}}_e\mathbf{F}_e^{-1})_s + \mathscr{L}_e^{-1} : (\tau\omega - \omega\tau). \tag{57}$$

The remaining part of the total strain rate is the plastic part

$$\mathbf{D}_p = [\mathbf{F}_e(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})\mathbf{F}_e^{-1}]_s - \mathscr{L}_e^{-1} : (\tau\omega - \omega\tau). \tag{58}$$

For example, if the material behavior is such that the Ilyushin postulate applies, the plastic strain rate $\mathbf{D}_p$ given by eqn (58) is governed by plastic potential and is normal to the corresponding yield surface in stress space. This follows by applying the Ilyushin postulate to certain finite or infinitesimal strain cycles, as shown by Hill (1968) and Hill and Rice (1973). The interpretation of the expressions analogous to eqns (57) and (58) that arise in the crystal plasticity studies has been given by Hill and Rice (1972), Hill and Havner (1982) and Asaro (1983).

The instantaneous elastic moduli tensor $\mathscr{L}_e$ defined by eqn (25), depends on the elastic deformation gradient $\mathbf{F}_e$, which is here defined relatively to isoclinic intermediate configuration. To determine $\mathbf{F}_e$, we proceed as follows. By the polar decomposition theorem, the elastic deformation gradient $\mathbf{F}_e$ can be expressed as $\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e$, where $\mathbf{V}_e$ is the elastic stretch and $\mathbf{R}_e$ is the elastic rotation tensor. Since the elastic strain energy ψ_e is an isotropic function of both $\mathbf{C}_e$ and $\hat{\mathscr{D}}$, it follows that

$$\psi_e(\mathbf{C}_e, \hat{\mathscr{D}}) = \psi_e(\mathbf{B}_e, \tilde{\mathscr{D}}). \tag{59}$$

In eqn (59), $\mathbf{B}_e = \mathbf{V}_e^2$ is the left Cauchy–Green elastic deformation tensor, while the rotation induced structural tensor $\tilde{\mathscr{D}}$ is defined by

$$\tilde{\mathscr{D}} = \mathbf{R}_e\hat{\mathscr{D}}\mathbf{R}_e^T = \tilde{\mathbf{n}} \otimes \tilde{\mathbf{n}}, \tag{60}$$

where

$$\tilde{\mathbf{n}} = \mathbf{R}_e\mathbf{n}_0. \tag{61}$$

The stress response (11) can consequently be written as

$$\tau = 2\mathbf{V}_e \frac{\partial\psi(\mathbf{B}_e, \tilde{\mathscr{D}})}{\partial\mathbf{B}_e}\mathbf{V}_e. \tag{62}$$

If the current state and the rotation $\mathbf{R}_e$ are known, eqn (62) gives a one-to-one relationship between the stress tensor τ and the elastic stretch tensor $\mathbf{V}_e = \mathbf{B}_e^{1/2}$. To obtain the elastic rotation $\mathbf{R}_e$, however, additional consideration is needed. For example, if on a certain physical basis an evolution equation for the spin $\mathbf{\Omega} = \dot{\mathbf{R}}_e\mathbf{R}_e^{-1}$ is constructed, the rotation $\mathbf{R}_e$ is obtained by the integration of

$$\dot{\mathbf{R}}_e = \mathbf{\Omega}\mathbf{R}_e. \tag{63}$$

Note that from eqn (61) the rate of change of the unit vector $\tilde{\mathbf{n}}$ is $\dot{\tilde{\mathbf{n}}} = \mathbf{\Omega}\tilde{\mathbf{n}}$. When the rotation $\mathbf{R}_e$ is determined, the elastic deformation tensor is calculated from $\mathbf{F}_e = \mathbf{V}_e\mathbf{R}_e$, where $\mathbf{V}_e$ is calculated from eqn (62). The direction of the axis of isotrophy $\mathbf{n}$ in the current configuration is then found from eqn (54).

8. CONCLUDING REMARKS

We have formulated in this paper the constitutive framework for the analysis of finite elastoplastic deformation in the presence of progressive degradation of the elastic material properties and corresponding damage. This has been accomplished by extending the model of the multiplicative decomposition of deformation gradient, which has previously been applied almost exclusively to the analysis of elastoplastic deformation of elastically isotropic materials, which remain isotropic during the plastic deformation processes. The exact kinematic and kinetic analysis of the finite deformation leads to the partition of the total strain rate into its elastic, damage and plastic parts. A general structure of the expression for the damage strain rate is derived, valid for introduced symmetric damage tensors of any order. The analysis of elastoplastic deformation of elastically anisotropic materials without damage is also presented, with application to transversely isotropic materials.

The presented work requires several extensions in order to complete the constitutive description of materials that undergo damage–elastoplastic deformation. The most immediate one is a development of the constitutive structure for the evolution equations for the appropriately specified damage variables. The coupling between plasticity and damage, elaboration on the structure of the yield and damage surfaces, existence of inelastic potentials and normality properties, are some of the associated questions also requiring further research. The valuable insight is already available from some of the previous related work, both in metal plasticity and rock and concrete inelasticity, such as Rice (1971), Rudnicki and Rice (1975), Nemat-Nasser (1983), Ortiz (1985), Ashby and Sammis (1990), Voyiadjis and Kattan (1992), Lubarda and Krajcinovic (1994b), etc.

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APPENDIX

The expressions for the rates of damage tensors given in Section 4, and for the damage strain rate given in Section 5, were derived by considering the damage tensors $\mathscr{D}$ to be the second-order symmetric tensors. We here show that the same derivation applies when the fourth- or higher-order damage tensors are introduced to adequately describe material degradation during a deformation process. Indeed, the fourth-order induced damage tensor $\hat{\mathscr{D}}$ in the intermediate configuration $\mathscr{P}_t$, corresponding to the fourth-order damage tensor $\mathscr{D}$ in the current configuration $\mathscr{B}_t$, has the covariant-type components

$$\hat{\mathscr{D}}_{ijkl} = |\mathbf{F}_\mathrm{e}|^{-m}(F^\mathrm{e}_{i\alpha})^\mathrm{T}(F^\mathrm{e}_{j\beta})^\mathrm{T}\mathscr{D}_{\alpha\beta\gamma\delta}F^\mathrm{e}_{\gamma k}F^\mathrm{e}_{\delta l}. \tag{A1}$$

The material time derivative of eqn (A1) is

$$\dot{\hat{\mathscr{D}}}_{ijkl} = |\mathbf{F}_\mathrm{e}|^{-m}(F^\mathrm{e}_{i\alpha})^\mathrm{T}(F^\mathrm{e}_{j\beta})^\mathrm{T}\overset{\bullet\mathrm{c}}{\mathscr{D}}_{\alpha\beta\gamma\delta}F^\mathrm{e}_{\gamma k}F^\mathrm{e}_{\delta l}, \tag{A2}$$

where $\overset{\bullet\mathrm{c}}{\mathscr{D}}$ represents a convected derivative of $\mathscr{D}$ relative to the velocity gradient $\dot{\mathbf{F}}_\mathrm{e}\mathbf{F}_\mathrm{e}^{-1}$. Substituting eqn (A2) into the second term on the right-hand side of the expression (23) therefore gives

$$\frac{\partial\tau}{\partial\hat{\mathscr{D}}}:\dot{\hat{\mathscr{D}}} = \frac{\partial\tau}{\partial\mathscr{D}}:\overset{\bullet\mathrm{c}}{\mathscr{D}}, \tag{A3}$$

which is the same structure as that of the previously derived expression (30), valid for the second-order damage tensors.

To derive an expression analogous to eqn (35), eliminate $\dot{\mathbf{F}}_\mathrm{e}\mathbf{F}_\mathrm{e}^{-1}$ from eqn (18) in terms of the total velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, to obtain

$$\begin{aligned}\dot{\hat{\mathscr{D}}}_{ijkl} = {} & |\mathbf{F}_\mathrm{e}|^{-m}(F^\mathrm{e}_{i\alpha})^\mathrm{T}(F^\mathrm{e}_{j\beta})^\mathrm{T}\overset{\bullet}{\mathscr{D}}_{\alpha\beta\gamma\delta}F^\mathrm{e}_{\gamma k}F^\mathrm{e}_{\delta l} + m\hat{\mathscr{D}}_{ijkl}\,\mathrm{tr}\,(\dot{\mathbf{F}}_\mathrm{p}\mathbf{F}_\mathrm{p}^{-1}) \\ & -\hat{\mathscr{D}}_{ijku}(\dot{\mathbf{F}}_\mathrm{p}\mathbf{F}_\mathrm{p}^{-1})_{ul} - \hat{\mathscr{D}}_{ijul}(\dot{\mathbf{F}}_\mathrm{p}\mathbf{F}_\mathrm{p}^{-1})_{uk} - (\dot{\mathbf{F}}_\mathrm{p}\mathbf{F}_\mathrm{p}^{-1})^\mathrm{T}_{ju}\hat{\mathscr{D}}_{iukl} - (\dot{\mathbf{F}}_\mathrm{p}\mathbf{F}_\mathrm{p}^{-1})^\mathrm{T}_{iu}\hat{\mathscr{D}}_{ujkl}.\end{aligned} \tag{A4}$$

In eqn (A4), $\overset{\bullet}{\mathscr{D}}$ denotes a convected derivative of the fourth-order tensor $\mathscr{D}$, relative to the velocity gradient $\mathbf{L}$, i.e. the derivative observed in the reference frame that deforms with the material in the current configuration $\mathscr{B}_t$. The components of this tensor are

$$\overset{\bullet}{\mathscr{D}}_{\alpha\beta\gamma\delta} = \dot{\mathscr{D}}_{\alpha\beta\gamma\delta} + \mathscr{D}_{\alpha\beta\gamma u}L_{u\delta} + \mathscr{D}_{\alpha\beta u\gamma}L_{u\gamma} + L^\mathrm{T}_{\beta u}\mathscr{D}_{\alpha u\gamma\delta} + L^\mathrm{T}_{\alpha u}\mathscr{D}_{u\beta\gamma\delta} - m\mathscr{D}_{\alpha\beta\gamma\delta}L_{uu}. \tag{A5}$$

Substitution of eqn (A4) into the second term on the right-hand side of the expression (23) then gives

$$\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathscr{D}}} : \dot{\hat{\mathscr{D}}} = \frac{\partial \boldsymbol{\tau}}{\partial \mathscr{D}} : \dot{\mathscr{D}} + \mathscr{A} : (\dot{\mathbf{F}}_{\mathrm{p}} \mathbf{F}_{\mathrm{p}}^{-1}), \tag{A6}$$

with the fourth-order tensor $\mathscr{A}$ defined by an expression similar to eqn (36b), corresponding to the second-order damage tensors. The expression (A6) coincides with the previously derived expression (35). An analogous derivation with the contravariant-type transformation leads to the same conclusion. Consequently, the derivation presented in Section 5 and the structure of the damage strain rate expressions (46) and (48) remain valid if the damage tensors are of the fourth order. The same is true for higher-order tensors, as well.